AD

TECHNICAL REPORT ARCCB-TR-95038

THRICE DIFFERENTIABLE AFFINE CONIC SPLINE INTERPOLATION

ROYCE W. SOANES

SEPTEMBER 1995

US ARMY ARMAMENT RESEARCH,
DEVELOPMENT AND ENGINEERING CENTER
CLOSE COMBAT ARMAMENTS CENTER
BENÉT LABORATORIES
WATERVLIET, N.Y. 12189-4050

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacturer(s) does not constitute an official indorsement or approval.

DESTRUCTION NOTICE

For classified documents, follow the procedures in DoD 5200.22-M, Industrial Security Manual, Section II-19 or DoD 5200.1-R, Information Security Program Regulation, Chapter IX.

For unclassified, limited documents, destroy by any method that will prevent disclosure of contents or reconstruction of the document.

For unclassified, unlimited documents, destroy when the report is no longer needed. Do not return it to the originator.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	September 1995	Final

4. TITLE AND SUBTITLE

THRICE DIFFERENTIABLE AFFINE
CONIC SPLINE INTERPOLATION

5. FUNDING NUMBERS

AMCMS: 6126.24.H180.0
PRON: 4A4A4FYB1ABJ

6. AUTHOR(S)

Royce W. Soanes

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

U.S. Army ARDEC
Benét Laboratories, AMSTA-AR-CCB-O
Watervliet, NY 12189-4050

8. PERFORMING ORGANIZATION REPORT NUMBER

ARCCB-TR-95038

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U.S. Army ARDEC
Close Combat Armaments Center
Picatinny Arsenal, NJ 07806-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

We present interpolating functions which have three orders of differentiability at each (convex) data point. These functions are defined as piecewise conics and are therefore guaranteed to be convex in the case of (strictly) convex data. The modifier "affine" refers to the fact that we make no use of Euclidean distance or angle in the discussion. We also include a discussion of numerical differentiation using conics. The nodal derivatives for the conic splines satisfy a locally quadrivariate quadratic system solved by Newton iteration--each iteration involving the solution of a pentadiagonal linear system. Initial values for Newton iteration are obtained by the aforementioned conic numerical differentiation. A discussion of numerical quadrature based on conic splines is also included, as well as a discussion of what we refer to as "sketched" interpolation, which makes use of the mathematical machinery behind conic differentiation and local C^3 conic splines. Sketched interpolation is more generally applicable than global C^3 conic splines are, as well as being computationally simpler, more flexible, and smoother in a local pointwise sense. This apparent increase of smoothness beyond C^3 is obtained through a process of re-sketching during the construction of the interpolant. Sketched interpolants reproduce conics with or without re-sketching. This is to say that if the discrete data comes from a conic, all the points of the sketched interpolant will lie on that conic.

14. SUBJECT TERMS

Spline, Conic, Interpolation, Affine Space, Affine Geometry, Numerical Differentiation, Numerical Quadrature, Abridged Notation

15. NUMBER OF PAGES

40

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

TABLE OF CONTENTS

INTRODUCTION 1

LINE MAPS 1

CONIC DERIVATIVES OR TANGENTS 2

PIECEWISE CONIC INTERPOLANTS 6

DERIVATIVE CONTINUITY 13

SYSTEM SOLUTION 14

NUMERICAL QUADRATURE 16

SKETCHED INTERPOLATION 22

RE-SKETCHING 25

FINAL NOTE ON TANGENTS 27

CONIC SECOND DERIVATIVES 29

DEALING WITH INFLECTION POINTS 31

REVIEW AND CONCLUSIONS 32

APPENDIX 33

TABLES

1. Line Map Indices 5

INTRODUCTION

We are mainly concerned with the smooth interpolation of convex data. We can interpolate nonconvex data, but it is obvious that we cannot get piecewise conics to have more than one continuous derivative at an inflection point. Inflection points may be few and far between anyway. Twice differentiable piecewise cubics are commonly used for interpolation but they carry no guarantee of shape preservation. They may be forced to preserve shape in the sense of convexity or monotonicity, but this may entail giving up an order of differentiability resulting in an interpolant of lower smoothness.

We make use of abridged notation that was popular among analytic geometers of the nineteenth century but which seems subsequently to have fallen into disuse among analysts and authors of modern texts on calculus and analytic geometry. The modern interest in practical computer-aided geometric design seems to have revived it somewhat, however.

LINE MAPS

The function

$$l(x,y) = ax+by+c$$

where a, b, and c are scalars and (x,y) is a vector in $\mathbf{R}^2$, represents one half of a general two-dimensional affine transformation. We refer to it as a line map for short. If we write

$$l(x,y) = 0$$

or simply

$$l=0$$

we have what is called abridged notation for the equation of a line in the plane. We have only one trivial theorem to prove regarding line maps. Suppose vector P is a real linear combination of other vectors Q.

$$P = \Sigma k_i Q_i$$

where

$$Q_i = (x_i, y_i)$$

Hence,

$$P = \Sigma k_i (x_i, y_i) = (\Sigma k_i x_i, \Sigma k_i y_i)$$

Now let

$$l(x,y) = ax + by + c$$

where

$$l(0,0) = c = l(0)$$

Now,

$$l(P) = a\Sigma k_i x_i + b\Sigma k_i y_i + c$$

$$= \Sigma k_i(ax_i + by_i) + c$$

$$= \Sigma k_i(l(Q_i) - c) + c$$

$$= \Sigma k_i l(Q_i) + c(1 - \Sigma k_i)$$

Therefore, we have

$$l(\Sigma k_i Q_i) = \Sigma k_i l(Q_i) + l(0)(1 - \Sigma k_i)$$

and we see that a line map is a linear map if $l(0)=0$ and that a line map at least behaves in a linear fashion if the coefficients of the linear combination sum to unity. In any case, we see how a line map operates on a linear combination of vectors.

CONIC DERIVATIVES OR TANGENTS

We can use line maps to conveniently represent conics in abridged notation also. Let P_n (i-2 $\leq n \leq i$+2) represent a sequence of five points in the plane which appears to derive from a single-valued convex function of a single variable. From the line

$$\frac{y - y_I}{x - x_I} = \frac{y_J - y_I}{x_J - x_I}$$

we define the line map

$$l_{IJ}(x,y) = (y - y_I)(x_J - x_I) - (x - x_I)(y_J - y_I)$$

and

$$l_{IJK} = l_{IJ}(P_K) = (y_K - y_I)(x_J - x_I) - (x_K - x_I)(y_J - y_I)$$

for arbitrary $I \neq J$ and K.

Now select a four-element subsequence of the aforementioned five-element sequence of points and let P_j be an element of the subsequence. Also let a and A be distinct line maps such that

$$a(x,y) = xa_x + ya_y + a_0,\ A(x,y) = xA_x + yA_y + A_0$$

and

$$a(P_j) = A(P_j) = 0$$

If P is a point on a conic through P_j, we have

$$P - P_j = (x - x_j, y - y_j) = (x - x_j)\left(1, \frac{y - y_j}{x - x_j}\right) = \alpha v$$

Now

$$a(P) = a(P_j + \alpha v) = a(P_j) + \alpha a(v) - \alpha a(0) = \alpha(a(v) - a(0))$$

and similarly

$$A(P) = \alpha(A(v) - A(0))$$

Hence

$$r(P) = \frac{a(P)}{A(P)} = \frac{a(v) - a_0}{A(v) - A_0}$$

If we now let

$$P \to P_j$$

we have

$$v \to (1, y_j'),\ r(P) \to r(P_j) = r_j$$

where

$$r_j = \frac{a(1, y_j') - a_0}{A(1, y_j') - A_0} = \frac{a_x + a_y y_j'}{A_x + A_y y_j'}$$

Solving the last equation for y_j', we get

$$y_j' = \frac{a_x - r_j A_x}{r_j A_y - a_y}$$

It is now only necessary to compute r_j for a conic in order to compute the derivative of or the tangent to the conic at P_j.

We must now define a connective graph relative to the four points of the subsequence. Define exactly four edges such that each vertex has exactly two incident edges. Define a line map corresponding to each edge such that line maps a and b are associated with nonadjacent edges and line maps A and B are also associated with nonadjacent edges. Of course, a and A must be associated with the vertex at which we seek the derivative. The product of line maps a and b is a degenerate hyperbola passing through the four points, as is the product of line maps A and B. Forming a linear combination of these line map products gives us "abridged notation" for a family or pencil of non-degenerate conics through these four points.

$$ab + \lambda AB = 0$$

Requiring this equation to hold for a fifth distinct point determines the parameter λ and the unique conic through five points. The point in the original sequence but not in the subsequence is called P_k. So, we compute

$$\lambda = -\frac{a(P_k)b(P_k)}{A(P_k)B(P_k)}$$

Now, dividing the conic equation by A, we have

$$rb + \lambda B = 0$$

or

$$r = -\frac{\lambda B}{b}$$

Therefore

$$r_j = -\frac{\lambda B(P_j)}{b(P_j)} = \frac{a(P_k)b(P_k)B(P_j)}{A(P_k)B(P_k)b(P_j)}$$

We select the edges of our connective graph so that the denominator of r_j is not zero. In addition, we select them so that this is the case even when one of the middle three points is collinear with its neighbors. All the graphs reflected in Table 1 have an hourglass edge pattern.

Table 1. Line Map Indices

j	k	a	b	A	B
i-2	i	i-2,i+2	i-1,i+1	i-2,i+1	i-1,i+2
i-1	i-2	i-1,i+1	i,i+2	i-1,i+2	i,i+1
i	i+2	i-1,i	i-2,i+1	i-2,i	i-1,i+1
i+1	i+2	i-1,i+1	i-2,i	i-2,i+1	i-1,i
i+2	i	i-2,i+2	i-1,i+1	i-1,i+2	i-2,i+1

Noting that

$$\partial_x l_{IJ} = y_I - y_J,\quad \partial_y l_{IJ} = x_J - x_I$$

we have

$$y_j' = \frac{y_{I_a} - y_{J_a} - r_j(y_{I_A} - y_{J_A})}{x_{I_a} - x_{J_a} - r_j(x_{I_A} - x_{J_A})}$$

$$r_j = \frac{l_{I_a,J_a,k}\, l_{I_b,J_b,k}\, l_{I_B,J_B,j}}{l_{I_A,J_A,k}\, l_{I_B,J_B,k}\, l_{I_b,J_b,j}}$$

where the I's and J's come from the table. Using the table, we can therefore simply write down the formulas for differentiating conics.

$$y_{i-2}' = \frac{y_{i-2} - y_{i+2} - r_{i-2}(y_{i-2} - y_{i+1})}{x_{i-2} - x_{i+2} - r_{i-2}(x_{i-2} - x_{i+1})}$$

$$r_{i-2} = \frac{l_{i-2,i+2,i}\, l_{i-1,i+1,i}\, l_{i-1,i+2,i-2}}{l_{i-2,i+1,i}\, l_{i-1,i+2,i}\, l_{i-1,i+1,i-2}}$$

$$y_{i-1}' = \frac{y_{i-1} - y_{i+1} - r_{i-1}(y_{i-1} - y_{i+2})}{x_{i-1} - x_{i+1} - r_{i-1}(x_{i-1} - x_{i+2})}$$

$$r_{i-1} = \frac{l_{i-1,i+1,i-2}\, l_{i,i+2,i-2}\, l_{i,i+1,i-1}}{l_{i-1,i+2,i-2}\, l_{i,i+1,i-2}\, l_{i,i+2,i-1}}$$

$$y_i' = \frac{y_{i-1} - y_i - r_i(y_{i-2} - y_i)}{x_{i-1} - x_i - r_i(x_{i-2} - x_i)}$$

$$r_i = \frac{l_{i-1,i,i+2}\, l_{i-2,i+1,i+2}\, l_{i-1,i+1,i}}{l_{i-2,i,i+2}\, l_{i-1,i+1,i+2}\, l_{i-2,i+1,i}}$$

$$y_{i+1}' = \frac{y_{i-1} - y_{i+1} - r_{i+1}(y_{i-2} - y_{i+1})}{x_{i-1} - x_{i+1} - r_{i+1}(x_{i-2} - x_{i+1})}$$

$$r_{i+1} = \frac{l_{i-1,i+1,i+2}\, l_{i-2,i,i+2}\, l_{i-1,i,i+1}}{l_{i-2,i+1,i+2}\, l_{i-1,i,i+2}\, l_{i-2,i,i+1}}$$

$$y_{i+2}' = \frac{y_{i-2} - y_{i+2} - r_{i+2}(y_{i-1} - y_{i+2})}{x_{i-2} - x_{i+2} - r_{i+2}(x_{i-1} - x_{i+2})}$$

$$r_{i+2} = \frac{l_{i-2,i+2,i}\, l_{i-1,i+1,i}\, l_{i-2,i+1,i+2}}{l_{i-1,i+2,i}\, l_{i-2,i+1,i}\, l_{i-1,i+1,i+2}}$$

These formulas for numerical differentiation based on conics can be used only when the data is locally convex. One must therefore suffer the inconvenience of testing the data for this property before using the formulas. Further caution is also needed in that although these formulas yield results superior to corresponding polynomial based formulas, polynomials at least carry with them the guarantee of single valuedness, whereas conics are globally bivalued. This being the case, one must be sure that the conic formulas yield derivative values of the proper sign at initial and final data points ($j=i$-2 or $j=i+2$). In any case, one can at least appreciate the value of abridged notation at this point.

PIECEWISE CONIC INTERPOLANTS

Consider four consecutive points on a convex curve--the first and last being fixed. Define line map a relative to the first two points, line map b relative to the last two points, line map A relative to the second and third points, and line map B relative to the first and last points. A family or pencil of conics through these four points is therefore given by

$$ab + \lambda AB = 0$$

Now, let the second point approach the first and the third point approach the last. In this case, $a=0$ and $b=0$ are lines tangent to the curve at the two fixed points. Line maps A and B coalesce into a single line map, say c. The family of conics interpolating the curve is then given by

$$ab+\lambda c^2=0$$

Denote the two fixed data points by P_i and P_{i+1}, the point of intersection of $a=0$ and $b=0$ by I, and the midpoint of P_iP_{i+1} by M. Of course,

$$a(P_i)=b(P_{i+1})=a(I)=b(I)=0$$

Define basis vectors u and v by

$$u=M-I=\frac{P_i+P_{i+1}}{2}-I$$

$$v=M-P_i=\frac{P_i+P_{i+1}}{2}-P_i=\frac{P_{i+1}-P_i}{2}$$

An arbitrary point P on our representative conic can therefore be expressed by

$$P=I+mu+nv=I+m\left(\frac{P_i+P_{i+1}}{2}-I\right)+n\left(\frac{P_{i+1}-P_i}{2}\right)$$

$$=(1-m)I+\left(\frac{m-n}{2}\right)P_i+\left(\frac{m+n}{2}\right)P_{i+1}$$

Note that

$$1-m+\frac{m-n}{2}+\frac{m+n}{2}=1$$

Therefore,

$$a(P)=\frac{m+n}{2}a(P_{i+1})$$

$$b(P)=\frac{m-n}{2}b(P_i)$$

Letting

$$K_i=a(P_{i+1})b(P_i)$$

we have

$$a(P)b(P)=\frac{K_i}{4}(m^2-n^2)$$

Now let point Q lie on line $c = 0$. But since P_i and P_{i+1} determine $c = 0$, we have

$$c(Q)=0,\ Q=(1-r)P_i+rP_{i+1}$$

Operating on Q with a and b, we have

$$a(Q)=ra(P_{i+1}),\ b(Q)=(1-r)b(P_i)$$

Eliminating parameter r, we get

$$r=\frac{a(Q)}{a(P_{i+1})}=\frac{b(P_i)-b(Q)}{b(P_i)}$$

or

$$a(Q)b(P_i)+b(Q)a(P_{i+1})-K_i=0$$

We therefore define $c(P)$ as

$$c(P)=a(P)b(P_i)+b(P)a(P_{i+1})-K_i$$

$$=\frac{m+n}{2}K_i+\frac{m-n}{2}K_i-K_i$$

$$=(m-1)K_i$$

From

$$a(P)b(P)+\frac{\lambda_i}{4}c(P)^2=0$$

we have

$$\frac{K_i}{4}(m^2-n^2)+\frac{\lambda_i}{4}(m-1)^2K_i^2=0$$

or equivalently

$$n^2=m^2+\lambda_iK_i(m-1)^2$$

This is therefore the equation of our family of conics in local (m,n) coordinates. Note from the definitions of u and v that we want $0 < m \leq 1$ and $-1 \leq n \leq 1$. Also, λ_iK_i must be negative. If $\lambda_iK_i < -1$ or $= -1$ or > -1, the conic is an ellipse, a parabola, or a hyperbola, respectively.

If we now differentiate the last equation three times with respect to m, we have

$$n\dot{n}=m+\lambda_i K_i(m-1)$$

$$\dot{n}^2+n\ddot{n}=1+\lambda_i K_i$$

$$3\dot{n}\ddot{n}+n\dddot{n}=0$$

from which we conclude that

$$n_i=-1,\ n_{i+1}=1$$

$$\dot{n}_i=-1,\ \dot{n}_{i+1}=1$$

$$\ddot{n}_i=-\lambda_i K_i,\ \ddot{n}_{i+1}=\lambda_i K_i$$

$$\dddot{n}_i=3\lambda_i K_i,\ \dddot{n}_{i+1}=-3\lambda_i K_i$$

In the following, let the prime denote differentiation with respect to x and the dot denote differentiation with respect to m as before. Recalling that

$$P=I+mu+nv$$

$$x=I_1+mu_1+nv_1$$

$$y=I_2+mu_2+nv_2$$

we have the following

$$\dot{x}=u_1+\dot{n}v_1$$

$$\dot{y}=u_2+\dot{n}v_2$$

$$y'=\frac{\dot{y}}{\dot{x}}=\frac{u_2+\dot{n}v_2}{u_1+\dot{n}v_1}$$

$$y''=\dot{x}^{-1}\frac{d}{dm}y'=\frac{(u_1v_2-u_2v_1)\ddot{n}}{(u_1+\dot{n}v_1)^3}$$

$$y'''=\dot{x}^{-1}\frac{d}{dm}y''=\frac{(u_1v_2-u_2v_1)(\dddot{n}(u_1+\dot{n}v_1)-3v_1\ddot{n}^2)}{(u_1+\dot{n}v_1)^5}$$

Abbreviating

$$u\wedge v=u_1v_2-u_2v_1$$

we have the first, second, and third nodal derivatives at the extremes of the subinterval:

$$y_i'=\frac{u_2-v_2}{u_1-v_1}$$

$$y_{i+1}'=\frac{u_2+v_2}{u_1+v_1}$$

$$y_i''=\frac{-\lambda_i K_i u\wedge v}{(u_1-v_1)^3}$$

$$y_{i+1}''=\frac{\lambda_i K_i u\wedge v}{(u_1+v_1)^3}$$

$$y_i'''=\frac{3\lambda_i K_i u\wedge v(u_1-v_1-v_1\lambda_i K_i)}{(u_1-v_1)^5}$$

$$y_{i+1}'''=\frac{-3\lambda_i K_i u\wedge v(u_1+v_1+v_1\lambda_i K_i)}{(u_1+v_1)^5}$$

Now we evaluate the various expressions involving the components of u and v in terms of the data and first derivatives. We leave out numerous algebraic steps for the sake of brevity. First,

$$u_2=v_2+y_i'(u_1-v_1)=-v_2+y_{i+1}'(u_1+v_1)$$

$$u_1=\frac{2v_2-v_1(y_i'+y_{i+1}')}{y_{i+1}'-y_i'}$$

Let

$$q_i=\frac{\Delta y_i}{\Delta x_i},\ A_i=y_i'-q_i,\ B_i=y_{i+1}'-q_i$$

Hence

$$u_1=\frac{v_1(2q_i-y_i'-y_{i+1}')}{\Delta y_i'}=\frac{-\Delta x_i(A_i+B_i)}{2\Delta y_i'}$$

$$u_1-v_1=\frac{-v_1(A_i+B_i)}{\Delta y_i'}-v_1=\frac{-B_i\Delta x_i}{\Delta y_i'}$$

$$u_1+v_1=\frac{-v_1(A_i+B_i)}{\Delta y_i'}+v_1=\frac{-A_i\Delta x_i}{\Delta y_i'}$$

Now

$$u_2=v_2+y_i'(u_1-v_1)=\frac{\Delta x_i}{2\Delta y_i'}(q_i\Delta y_i'-2B_iy_i')$$

but

$$q_i\Delta y_i'-2B_iy_i'=-A_iy_{i+1}'-B_iy_i'$$

Hence

$$u_2=\frac{-\Delta x_i(A_iy_{i+1}'+B_iy_i')}{2\Delta y_i'}$$

Also,

$$u_1v_2-u_2v_1=\frac{-q_i\Delta x_i^2(A_i+B_i)}{4\Delta y_i'}+\frac{\Delta x_i^2(A_iy_{i+1}'+B_iy_i')}{4\Delta y_i'}=\frac{A_iB_i\Delta x_i^2}{2\Delta y_i'}$$

In addition,

$$I=M-u=P_i+v-u=P_{i+1}-v-u$$

Hence

$$I_1=x_i+\frac{B_i\Delta x_i}{\Delta y_i'}=x_{i+1}+\frac{A_i\Delta x_i}{\Delta y_i'}$$

But also,

$$\frac{I_2-y_i}{I_1-x_i}=y_i',\ \frac{I_2-y_{i+1}}{I_1-x_{i+1}}=y_{i+1}'$$

Therefore, we have

$$I_2=y_i+\frac{B_iy_i'\Delta x_i}{\Delta y_i'}=y_{i+1}+\frac{A_iy_{i+1}'\Delta x_i}{\Delta y_i'}$$

We now define a and b explicitly and compute K_i.

$$a(x,y)=(y-y_i)(I_1-x_i)-(x-x_i)(I_2-y_i)$$

$$b(x,y)=(y-y_{i+1})(I_1-x_{i+1})-(x-x_{i+1})(I_2-y_{i+1})$$

Thus

$$a(P_{i+1})=\Delta y_i(I_1-x_i)-\Delta x_i(I_2-y_i)=\frac{-A_iB_i\Delta x_i^2}{\Delta y_i'}$$

$$b(P_i)=-\Delta y_i(I_1-x_{i+1})+\Delta x_i(I_2-y_{i+1})=\frac{A_iB_i\Delta x_i^2}{\Delta y_i'}$$

$$K_i=a(P_{i+1})b(P_i)=\frac{-A_i^2B_i^2\Delta x_i^4}{\Delta y_i'^2}<0$$

We see here that λ_i must be positive. Now we are prepared to evaluate the second and third derivatives at the ends of the subinterval in terms of the (x,y) data, the nodal derivatives, and λ_i.

$$y_i''=\frac{-\lambda_iK_iu\wedge v}{(u_1-v_1)^3}=-\tfrac{1}{2}\lambda_iA_i^3\Delta x_i^3$$

$$y_{i+1}''=\frac{\lambda_iK_iu\wedge v}{(u_1+v_1)^3}=\tfrac{1}{2}\lambda_iB_i^3\Delta x_i^3$$

$$y_i'''=\frac{3\lambda_iK_iu\wedge v(u_1-v_1-v_1\lambda_iK_i)}{(u_1-v_1)^5}=\frac{-3\lambda_iA_i^3\Delta x_i^2}{2B_i}(\Delta y_i'-\tfrac{1}{2}\lambda_iA_i^2B_i\Delta x_i^4)$$

$$y_{i+1}'''=\frac{-3\lambda_iK_iu\wedge v(u_1+v_1+v_1\lambda_iK_i)}{(u_1+v_1)^5}=\frac{3\lambda_iB_i^3\Delta x_i^2}{2A_i}(\Delta y_i'+\tfrac{1}{2}\lambda_iA_iB_i^2\Delta x_i^4)$$

DERIVATIVE CONTINUITY

In this section, we use the conditions for continuity of the second and third derivatives to derive a system of equations solvable for the nodal derivatives of a C^3 piecewise conic. Enforcing continuity of the second derivative at x_i gives us

$$\lambda_{i-1}B_{i-1}^3\Delta x_{i-1}^3 = -\lambda_i A_i^3 \Delta x_i^3$$

We notice here that if we were to assign all the nodal derivatives by conic differentiation, some other local method, or even arbitrarily, we would only need to assign λ on one subinterval and use the previous continuity condition recursively to obtain all the other λ's for a C^2 piecewise conic. There is definitely something to be said for interpolants of lower smoothness, especially in the case of piecewise conics. For instance, we obviously do not want any more than C^1 smoothness to link a curve with a line in the case of conics, since only degenerate conics can be flat. But, since we have advertised the ultimate in smoothness for piecewise conics, we continue in this direction. Enforcing continuity of the third derivative at x_i gives us

$$\frac{\lambda_{i-1}B_{i-1}^3\Delta x_{i-1}^2}{A_{i-1}}\left(\Delta y_{i-1}' + \frac{1}{2}\lambda_{i-1}A_{i-1}B_{i-1}^2\Delta x_{i-1}^4\right) = \frac{-\lambda_i A_i^3 \Delta x_i^2}{B_i}\left(\Delta y_i' - \frac{1}{2}\lambda_i A_i^2 B_i \Delta x_i^4\right)$$

Now, we use the second derivative continuity condition to solve for λ_{i-1}, substitute the resulting expression into the third derivative continuity condition, simplify, and finally solve the resulting equation for λ_i.

$$\lambda_{i-1} = -\frac{\lambda_i A_i^3 \Delta x_i^3}{B_{i-1}^3 \Delta x_{i-1}^3}$$

$$\lambda_i = \frac{2B_{i-1}(A_{i-1}\Delta y_i' \Delta x_{i-1} - B_i \Delta y_{i-1}' \Delta x_i)}{A_{i-1}A_i^2 B_i \Delta x_i^4 \Delta x_{i-1} \Delta q_{i-1}}$$

The algebra involved in the last step is mildly tedious, but there are numerous fortuitous cancellations. Now, enforcing continuity of the second and third derivatives at x_{i+1} gives us

$$\lambda_i B_i^3 \Delta x_i^3 = -\lambda_{i+1} A_{i+1}^3 \Delta x_{i+1}^3$$

$$\frac{\lambda_i B_i^3 \Delta x_i^2}{A_i}\left(\Delta y_i' + \frac{1}{2}\lambda_i A_i B_i^2 \Delta x_i^4\right) = -\frac{\lambda_{i+1} A_{i+1}^3 \Delta x_{i+1}^2}{B_{i+1}}\left(\Delta y_{i+1}' - \frac{1}{2}\lambda_{i+1} A_{i+1}^2 B_{i+1} \Delta x_{i+1}^4\right)$$

In a similar manner, we now eliminate λ_{i+1}, using the second derivative continuity condition, and solve for λ_i in the condition for third derivative continuity, getting

$$\lambda_i = \frac{2A_{i+1}(B_{i+1}\Delta y_i^{\prime}\Delta x_{i+1} - A_i\Delta y_{i+1}^{\prime}\Delta x_i)}{A_i B_i^2 B_{i+1}\Delta x_i^4 \Delta x_{i+1}\Delta q_i}$$

Now, we simply equate the two expressions defining λ_i, and clear of fractions, getting

$$A_{i-1}A_iA_{i+1}\Delta q_{i-1}\Delta x_{i-1}(B_{i+1}\Delta y_i^{\prime}\Delta x_{i+1} - A_i\Delta y_{i+1}^{\prime}\Delta x_i)$$
$$-B_{i-1}B_iB_{i+1}\Delta q_i\Delta x_{i+1}(A_{i-1}\Delta y_i^{\prime}\Delta x_{i-1} - B_i\Delta y_{i-1}^{\prime}\Delta x_i)=0$$

This equation, which is the main result of this section, is a necessary condition for continuity of the second and third derivatives at two neighboring points in terms of the data and first derivatives at four neighboring points. It behaves quadratically in the two inner derivatives and linearly in the two outer derivatives. We therefore refer to it as a component of a locally quadrivariate quadratic system. It behooves us at this point to make a preliminary examination of the system. We can write a component equation only for interior subintervals, because the quadrivariate nature of each component equation forces us to exclude the first and last subintervals. Assuming that we have n points of data, we can therefore write only n-3 component equations. This leaves us free to specify derivatives at the first and last points and one interior point by either conic differentiation or other means. If we use conic differentiation to specify the three derivatives and the data comes from a conic, the conic is preserved. This gives us a numerical check on all the algebra. The need to set one internal derivative causes the linear systems involved in Newton iteration to be pentadiagonal.

SYSTEM SOLUTION

This locally quadrivariate quadratic system is solved using Newton iteration. Initial values for the unknown nodal derivatives can be computed using conic differentiation. We define the following expressions for the sake of brevity.

$$\alpha_i = A_{i-1}A_iA_{i+1}\Delta q_{i-1}\Delta x_{i-1}$$
$$\beta_i = B_{i-1}B_iB_{i+1}\Delta q_i\Delta x_{i+1}$$
$$\gamma_i = B_{i+1}\Delta y_i^{\prime}\Delta x_{i+1} - A_i\Delta y_{i+1}^{\prime}\Delta x_i$$
$$\delta_i = A_{i-1}\Delta y_i^{\prime}\Delta x_{i-1} - B_i\Delta y_{i-1}^{\prime}\Delta x_i$$

Our system component equations are therefore

$$f_i = \alpha_i\gamma_i - \beta_i\delta_i = 0$$

The system we wish to solve is therefore

$$f(y^{\prime})=0$$

augmented by three trivial equations. A Newton iteration is accomplished by solving the

linear(ized) system (augmented)

$$df(y') = -f(y')$$

for dy' and incrementing y' by dy'. To write this system, we must now compute the partial derivatives of the f's. Abbreviate

$$\partial_k = \frac{\partial}{\partial y_k'}$$

Compute

$$\partial_{i-1}\alpha_i = A_i A_{i+1} \Delta q_{i-1} \Delta x_{i-1}$$
$$\partial_i \alpha_i = A_{i-1} A_{i+1} \Delta q_{i-1} \Delta x_{i-1}$$
$$\partial_{i+1}\alpha_i = A_{i-1} A_i \Delta q_{i-1} \Delta x_{i-1}$$
$$\partial_{i+2}\alpha_i = 0$$

$$\partial_{i-1}\beta_i = 0$$
$$\partial_i \beta_i = B_i B_{i+1} \Delta q_i \Delta x_{i+1}$$
$$\partial_{i+1}\beta_i = B_{i-1} B_{i+1} \Delta q_i \Delta x_{i+1}$$
$$\partial_{i+2}\beta_i = B_{i-1} B_i \Delta q_i \Delta x_{i+1}$$

$$\partial_{i-1}\gamma_i = 0$$
$$\partial_i \gamma_i = -B_{i+1}\Delta x_{i+1} - \Delta y_{i+1}' \Delta x_i$$
$$\partial_{i+1}\gamma_i = B_{i+1}\Delta x_{i+1} + A_i \Delta x_i$$
$$\partial_{i+2}\gamma_i = \Delta y_i' \Delta x_{i+1} - A_i \Delta x_i$$

$$\partial_{i-1}\delta_i = \Delta y_i' \Delta x_{i-1} + B_i \Delta x_i$$
$$\partial_i \delta_i = -A_{i-1}\Delta x_{i-1} - B_i \Delta x_i$$
$$\partial_{i+1}\delta_i = A_{i-1}\Delta x_{i-1} - \Delta y_{i-1}' \Delta x_i$$
$$\partial_{i+2}\delta_i = 0$$

$$\partial_j f_i = \alpha_i \partial_j \gamma_i + \gamma_i \partial_j \alpha_i - \beta_i \partial_j \delta_i - \delta_i \partial_j \beta_i$$

The linear system in which these expressions are used is given by

$$
\begin{gathered}
1\cdot dy_0'=0\\
\partial_0 f_1 dy_0'+\partial_1 f_1 dy_1'+\partial_2 f_1 dy_2'+\partial_3 f_1 dy_3'=-f_1\\
\vdots\\
\partial_{k-2} f_{k-1} dy_{k-2}'+\partial_{k-1} f_{k-1} dy_{k-1}'+\partial_k f_{k-1} dy_k'+\partial_{k+1} f_{k-1} dy_{k+1}'=-f_{k-1}\\
1\cdot dy_k'=0\\
\partial_{k-1} f_k dy_{k-1}'+\partial_k f_k dy_k'+\partial_{k+1} f_k dy_{k+1}'+\partial_{k+2} f_k dy_{k+2}'=-f_k\\
\vdots\\
\partial_{n-4} f_{n-3} dy_{n-4}'+\partial_{n-3} f_{n-3} dy_{n-3}'+\partial_{n-2} f_{n-3} dy_{n-2}'+\partial_{n-1} f_{n-3} dy_{n-1}'=-f_{n-3}\\
1\cdot dy_{n-1}'=0
\end{gathered}
$$

Note that in the "f_{k-1}" equation, the diagonal element of the matrix is associated with dy_{k-1}' and there are two nonzero terms to the right. Also note that in the "f_k" equation, the diagonal element of the matrix is associated with dy_{k+1}' and there are two nonzero terms to the left. The linear Newton iteration system is therefore pentadiagonal.

NUMERICAL QUADRATURE

We now consider the use of conic splines for numerical integration using previously obtained material. Considering a single subinterval, let A be the area between the conic and the chord. Let P be a point on the conic and Q be a point on the chord such that P-Q is parallel to u. Our element of area is therefore given by

$$dA=dQ\wedge(P-Q)$$

Now Q is given by

$$Q=P_i+\rho(P_{i+1}-P_i)=P_i+2\rho v$$

and P-Q is given by

$$
\begin{aligned}
P-Q&=I+mu+nv-P_i-2\rho v\\
&=P_i+v-u+mu+nv-P_i-2\rho v\\
&=(m-1)u+(1+n-2\rho)v
\end{aligned}
$$

But since P-Q is parallel to u, we have

$$u\wedge(P-Q)=0=(1+n-2\rho)u\wedge v$$

and

$$\rho=\frac{n+1}{2}$$

Hence,

$$dQ=v\,dn$$
$$P-Q=(m-1)u$$
$$dA=v\,dn\wedge(m-1)u$$
$$=u\wedge v(1-m)dn$$

and finally,

$$A=u\wedge v\int_{-1}^{1}(1-m)dn$$

Defining

$$-s^2=\lambda_i K_i$$

and recalling that

$$n^2=m^2-s^2(m-1)^2$$

we can solve this quadratic equation for 1-*m*, getting

$$1-m=\frac{1-\sqrt{1-(1-s^2)(1-n^2)}}{1-s^2}$$

If we were now to plug this expression into the previous integral defining A, we would be able to obtain an analytic expression for the integral in terms of simple algebraic and transcendental functions. Unfortunately, the resulting expressions are asymptotically unrobust in the numerical sense. By this we mean that as the integration mesh becomes dense in the limit, s approaches unity and produces numerically indeterminate forms. The fact that s approaches unity in the limit is fairly easy to prove, but the algebra is somewhat tedious, so we omit the proof here. It might be expected that conic splines would become (oriented) parabolic splines in the limit anyway. We therefore opt for a quite robust infinite series development instead. Consider the function

$$f(x)=\sqrt{1-x}$$

where x is close to zero. Differentiating successively, we have

$$f'(x) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$$

$$\vdots$$

$$f^{(k)}(x) = -\frac{3\cdot 5\cdot 7\cdots(2k-3)}{2^k}(1-x)^{-\frac{2k-1}{2}}$$

$$= -\frac{(2k-3)!}{(k-2)!\,2^{2k-2}}(1-x)^{-\frac{2k-1}{2}}$$

and therefore

$$\frac{f^{(k)}(0)}{k!} = -\frac{(2k-3)!}{k!(k-2)!\,2^{2k-2}} \qquad (k>1)$$

We therefore have the Taylor series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

$$= \sqrt{1-x} = 1 - \frac{x}{2} - \sum_{k=2}^{\infty} \frac{(2k-3)!\,x^k}{k!(k-2)!\,2^{2k+2}}$$

$$= 1 - \frac{x}{2} - \sum_{k=0}^{\infty} \frac{(2k+1)!\,x^{k+2}}{k!(k+2)!\,2^{2k+2}}$$

from which we get

$$1 - \sqrt{1-x} = \frac{x}{2} + \sum_{k=0}^{\infty} \frac{(2k+1)!\,x^{k+2}}{k!(k+2)!\,2^{2k+2}}$$

Therefore,

$$1 - \sqrt{1-(1-s^2)(1-n^2)} = \frac{(1-s^2)(1-n^2)}{2} + \sum_{k=0}^{\infty} \frac{(2k+1)!(1-s^2)^{k+2}(1-n^2)^{k+2}}{k!(k+2)!\,2^{2k+2}}$$

and finally,

$$1 - m = \frac{1-n^2}{2} + \sum_{k=0}^{\infty} \frac{(2k+1)!(1-s^2)^{k+1}(1-n^2)^{k+2}}{k!(k+2)!\,2^{2k+2}}$$

This series is obviously very rapidly convergent and robust in the limit as s approaches unity. We must now integrate this series term by term with respect to n.

$$\int_{-1}^{1}(1-m)dn = \int_{-1}^{1}\frac{1-n^2}{2}dn + \sum_{k=0}^{\infty}\frac{(2k+1)!(1-s^2)^{k+1}}{k!(k+2)!\,2^{2k+2}}\int_{-1}^{1}(1-n^2)^{k+2}dn$$
$$= \int_{0}^{1}1-n^2dn + \sum_{k=0}^{\infty}\frac{(2k+1)!(1-s^2)^{k+1}}{k!(k+2)!\,2^{2k+1}}\int_{0}^{1}(1-n^2)^{k+2}dn$$

Now, we must consider computing an integral of the form

$$\int_{0}^{1}(1-x^2)^p dx$$

We do this by successive integration by parts, using the formula

$$\int_{a}^{b}f(x)g(x)dx = f(b)\int_{a}^{b}g(x)dx - \int_{a}^{b}f'(x)\int_{a}^{x}g(t)dt\,dx$$

to finally obtain

$$\int_{0}^{1}(1-x^2)^p dx = \frac{2^{2p}(p!)^2}{(2p+1)!}$$

from which we get

$$\int_{0}^{1}(1-n^2)^{k+2}dn = \frac{2^{2k+4}((k+2)!)^2}{(2k+5)!}$$
$$\int_{0}^{1}1-n^2dn = \frac{4}{3!} = \frac{2}{3}$$

We therefore have

$$\int_{-1}^{1}1-m\,dn = \frac{2}{3} + \sum_{k=0}^{\infty}\frac{(2k+1)!(1-s^2)^{k+1}}{k!(k+2)!\,2^{2k+1}}\,\frac{2^{2k+4}((k+2)!)^2}{k!(2k+5)!}$$

which easily simplifies to

$$\int_{-1}^{1}1-m\,dn = 2\left(\frac{1}{3} + \sum_{k=0}^{\infty}\frac{(1-s^2)^{k+1}}{(2k+3)(2k+5)}\right)$$

But,

$$u \wedge v = \frac{A_i B_i \Delta x_i^2}{2\Delta y_i'}$$

Hence, we finally have

$$A = \frac{A_i B_i \Delta x_i^2}{\Delta y_i'}\left(\frac{1}{3} + \sum_{k=0}^{\infty} \frac{(1+\lambda_i K_i)^{k+1}}{(2k+3)(2k+5)}\right)$$

Now, if we simply add in the trapezoidal component of the integral, we have the following general quadrature formula

$$\int_{x_i}^{x_{i+1}} y(x)dx = \frac{\Delta x_i}{2}(y_i + y_{i+1}) + \frac{A_i B_i \Delta x_i^2}{\Delta y_i'}\left(\frac{1}{3} + \sum_{k=0}^{\infty} \frac{(1+\lambda_i K_i)^{k+1}}{(2k+3)(2k+5)}\right)$$

Recalling that

$$\lambda_i = \frac{2B_{i-1}(A_{i-1}\Delta y_i' \Delta x_{i-1} - B_i \Delta y_{i-1}' \Delta x_i)}{A_{i-1} A_i^2 B_i \Delta x_i^4 \Delta x_{i-1} \Delta q_{i-1}}$$
$$= \frac{2A_{i+1}(B_{i+1}\Delta y_i' \Delta x_{i+1} - A_i \Delta y_{i+1}' \Delta x_i)}{A_i B_i^2 B_{i+1} \Delta x_i^4 \Delta x_{i+1} \Delta q_i}$$

and that

$$K_i = -\frac{A_i^2 B_i^2 \Delta x_i^4}{\Delta y_i'^{\,2}}$$

we easily have that

$$\lambda_i K_i = \frac{2B_{i-1}B_i(B_i \Delta y_{i-1}' \Delta x_i - A_{i-1}\Delta y_i' \Delta x_{i-1})}{A_{i-1}\Delta x_{i-1}\Delta q_{i-1}\Delta y_i'^{\,2}}$$
$$= \frac{2A_i A_{i+1}(A_i \Delta y_{i+1}' \Delta x_i - B_{i+1}\Delta y_i' \Delta x_{i+1})}{B_{i+1}\Delta x_{i+1}\Delta q_i \Delta y_i'^{\,2}}$$

We have three different options with regard to using this quadrature formula. The first and most accurate option is to compute the nodal derivatives for a C^3 conic spline and then use the quadrature formula in conjunction with either of the two previous formulas. The second and less accurate option is to simply obtain the nodal derivatives by conic differentiation and use the average value provided by the two previous formulas. The third and least accurate option is

again to use conic differentiation, but then to simply discard the infinite series altogether--especially if the mesh is dense. This last option, though least accurate, is quite viable, since it requires the least computation and integrates any oriented parabola exactly. In fact, these quadrature formulas are sufficiently robust to accommodate infinite derivatives, provided that the numerators and denominators of the conic derivatives are calculated separately (tangent vectors). Since

$$A_i = y_i' - q_i = \frac{dy_i}{dx_i} - \frac{\Delta y_i}{\Delta x_i} = \frac{\Delta x_i dy_i - \Delta y_i dx_i}{\Delta x_i dx_i} = \frac{\Delta_i \wedge d_i}{\Delta x_i dx_i},$$

$$B_i = y_{i+1}' - q_i = \frac{dy_{i+1}}{dx_{i+1}} - \frac{\Delta y_i}{\Delta x_i} = \frac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}},$$

$$\Delta y_i' = y_{i+1}' - y_i' = \frac{dy_{i+1}}{dx_{i+1}} - \frac{dy_i}{dx_i} = \frac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}},$$

and

$$\frac{A_i B_i \Delta x_i^2}{\Delta y_i'} = \frac{\dfrac{\Delta_i \wedge d_i}{\Delta x_i dx_i}\dfrac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}}\Delta x_i^2}{\dfrac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}}} = \frac{(d_i \wedge \Delta_i)(d_{i+1} \wedge \Delta_i)}{d_i \wedge d_{i+1}}$$

The quadrature formula in this case is easily seen to be

$$\int_{x_i}^{x_{i+1}} y(x)dx = \frac{\Delta x_i}{2}(y_i + y_{i+1}) + \frac{1}{3}\frac{(d_i \wedge \Delta_i)(d_{i+1} \wedge \Delta_i)}{d_i \wedge d_{i+1}}$$

It is instructive to compute

$$\int_0^1 \sqrt{x}dx = \frac{2}{3}$$

using this quadrature formula and

$$y_i=0,\ y_{i+1}=1,\ \Delta x_i=1,\ \Delta y_i=1$$
$$dx_i=0,\ dy_i\neq 0,\ dx_{i+1}=2,\ dy_{i+1}=1$$

SKETCHED INTERPOLATION

By sketching, we mean: starting with at least five data points (or fewer, if initial tangents are specified) and during multiple passes over the accumulated data, computing the nodal tangents by conic differentiation and inserting exactly one point between each two data points on each pass using the previous expressions for $\lambda_i K_i$. We can thereby produce a set of discrete points as dense as we wish to represent the curve. If the original points lie on a conic, the generated points will also lie exactly on that conic. In addition, this method of sketching accommodates itself not only to single-valued data but to double-valued data and closed curves as well, because the expressions for $\lambda_i K_i$ may be cast into another more robust form. First, we recall that

$$\lambda_i K_i = \frac{2B_{i-1}B_i(B_i \Delta y'_{i-1}\Delta x_i - A_{i-1}\Delta y'_i \Delta x_{i-1})}{A_{i-1}\Delta x_{i-1}\Delta q_{i-1}\Delta y_i'^{\,2}}$$

$$\Delta_i = (\Delta x_i, \Delta y_i),\ d_i = (dx_i, dy_i),\ (a,b)\wedge(c,d) = ad - bc$$

Then we have

$$\Delta q_i = q_{i+1} - q_i = \frac{\Delta y_{i+1}}{\Delta x_{i+1}} - \frac{\Delta y_i}{\Delta x_i} = \frac{\Delta_i \wedge \Delta_{i+1}}{\Delta x_i \Delta x_{i+1}}$$

We also have

$$\begin{aligned} & B_i \Delta y'_{i-1}\Delta x_i - A_{i-1}\Delta y'_i \Delta x_{i-1} \\ &= \frac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}} \Delta x_i \frac{d_{i-1}\wedge d_i}{dx_{i-1}dx_i} - \frac{\Delta_{i-1}\wedge d_{i-1}}{\Delta x_{i-1}dx_{i-1}} \Delta x_{i-1} \frac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}} \\ &= \frac{1}{dx_{i-1}dx_i dx_{i+1}}\left((\Delta_i \wedge d_{i+1})(d_{i-1}\wedge d_i) - (\Delta_{i-1}\wedge d_{i-1})(d_i \wedge d_{i+1})\right) \end{aligned}$$

and

$$\begin{aligned} \frac{B_{i-1}B_i}{A_{i-1}\Delta x_{i-1}\Delta q_{i-1}\Delta y_i'^{\,2}} &= \frac{\dfrac{\Delta_{i-1}\wedge d_i}{\Delta x_{i-1}dx_i}\,\dfrac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}}}{\dfrac{\Delta_{i-1}\wedge d_{i-1}}{\Delta x_{i-1}dx_{i-1}}\Delta x_{i-1}\dfrac{\Delta_{i-1}\wedge\Delta_i}{\Delta x_{i-1}\Delta x_i}\left(\dfrac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}}\right)^2} \\ &= dx_{i-1}dx_i dx_{i+1}\frac{(\Delta_{i-1}\wedge d_i)(\Delta_i \wedge d_{i+1})}{(\Delta_{i-1}\wedge d_{i-1})(\Delta_{i-1}\wedge\Delta_i)(d_i \wedge d_{i+1})^2} \end{aligned}$$

Then, multiplying the appropriate quantities, we finally have

$$\lambda_i K_i = \frac{2(\Delta_{i-1}\wedge d_i)(\Delta_i \wedge d_{i+1})}{(\Delta_{i-1}\wedge d_{i-1})(\Delta_{i-1}\wedge\Delta_i)(d_i\wedge d_{i+1})^2}((\Delta_i\wedge d_{i+1})(d_{i-1}\wedge d_i)-(\Delta_{i-1}\wedge d_{i-1})(d_i\wedge d_{i+1}))$$

$$= \frac{2(\Delta_{i-1}\wedge d_i)(\Delta_i\wedge d_{i+1})}{(\Delta_{i-1}\wedge\Delta_i)(d_i\wedge d_{i+1})}\left(\frac{(\Delta_i\wedge d_{i+1})(d_{i-1}\wedge d_i)}{(\Delta_{i-1}\wedge d_{i-1})(d_i\wedge d_{i+1})}-1\right)$$

This is the most robust form for computing a value of $\lambda_i K_i$ which ensures continuity of the second and third derivatives on the left. This formula alone would be used for the last subinterval. Now recall also that

$$\lambda_i K_i = \frac{2A_i A_{i+1}(A_i \Delta y'_{i+1}\Delta x_i - B_{i+1}\Delta y'_i \Delta x_{i+1})}{B_{i+1}\Delta x_{i+1}\Delta q_i \Delta y_i'^{\,2}}$$

We then have that

$$A_i \Delta y'_{i+1}\Delta x_i - B_{i+1}\Delta y'_i \Delta x_{i+1}$$

$$= \frac{\Delta_i\wedge d_i}{\Delta x_i dx_i}\Delta x_i \frac{d_{i+1}\wedge d_{i+2}}{dx_{i+1}dx_{i+2}} - \frac{\Delta_{i+1}\wedge d_{i+2}}{\Delta x_{i+1}dx_{i+2}}\Delta x_{i+1}\frac{d_i\wedge d_{i+1}}{dx_i dx_{i+1}}$$

$$= \frac{1}{dx_i dx_{i+1}dx_{i+2}}((\Delta_i\wedge d_i)(d_{i+1}\wedge d_{i+2})-(\Delta_{i+1}\wedge d_{i+2})(d_i\wedge d_{i+1}))$$

and

$$\frac{A_i A_{i+1}}{B_{i+1}\Delta x_{i+1}\Delta q_i \Delta y_i'^{\,2}} = \frac{\dfrac{\Delta_i\wedge d_i}{\Delta x_i dx_i}\dfrac{\Delta_{i+1}\wedge d_{i+1}}{\Delta x_{i+1}dx_{i+1}}}{\dfrac{\Delta_{i+1}\wedge d_{i+2}}{\Delta x_{i+1}dx_{i+2}}\Delta x_{i+1}\dfrac{\Delta_i\wedge\Delta_{i+1}}{\Delta x_i \Delta x_{i+1}}\left(\dfrac{d_i\wedge d_{i+1}}{dx_i dx_{i+1}}\right)^2}$$

$$= dx_i dx_{i+1} dx_{i+2}\frac{(\Delta_i\wedge d_i)(\Delta_{i+1}\wedge d_{i+1})}{(\Delta_{i+1}\wedge d_{i+2})(\Delta_i\wedge\Delta_{i+1})(d_i\wedge d_{i+1})^2}$$

Multiplying the appropriate quantities, we then have

$$\lambda_i K_i = \frac{2(\Delta_i \wedge d_i)(\Delta_{i+1} \wedge d_{i+1})}{(\Delta_{i+1} \wedge d_{i+2})(\Delta_i \wedge \Delta_{i+1})(d_i \wedge d_{i+1})^2}((\Delta_i \wedge d_i)(d_{i+1} \wedge d_{i+2}) - (\Delta_{i+1} \wedge d_{i+2})(d_i \wedge d_{i+1}))$$

$$= \frac{2(\Delta_i \wedge d_i)(\Delta_{i+1} \wedge d_{i+1})}{(\Delta_i \wedge \Delta_{i+1})(d_i \wedge d_{i+1})}\left(\frac{(\Delta_i \wedge d_i)(d_{i+1} \wedge d_{i+2})}{(\Delta_{i+1} \wedge d_{i+2})(d_i \wedge d_{i+1})} - 1\right)$$

This is the most robust form for computing a value of $\lambda_i K_i$ which ensures continuity of the second and third derivatives on the right. This formula alone would be used for the first subinterval. For interior subintervals, we compute a value for $\lambda_i K_i$ for the left and right C^3 three point local conic splines, insert a point on each, and take the midpoint of these. All we need do at this point is to show how to compute a single point on the desired curve. Letting P be this point, recalling that

$$P = I + mu + nv$$
$$n^2 = m^2 + \lambda_i K_i (m-1)^2 = m^2 - s^2(m-1)^2$$

and letting $n=0$ gives us

$$P = I + mu$$
$$m = \frac{s}{1+s}$$

But from

$$I = M - u$$

we have that

$$P = M - (1-m)u = M - \frac{u}{1+s}$$

Now, it is only necessary to compute the components of u. Recalling that

$$u_1 = -\frac{\Delta x_i (A_i + B_i)}{2\Delta y_i'}$$
$$u_2 = -\frac{\Delta x_i (A_i y_{i+1}' + B_i y_i')}{2\Delta y_i'}$$

we have almost immediately that

$$u_1 = -\frac{\Delta x_i\left(\frac{\Delta_i \wedge d_i}{\Delta x_i dx_i} + \frac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}}\right)}{2\frac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}}} = \frac{(d_i \wedge \Delta_i)dx_{i+1} + (d_{i+1} \wedge \Delta_i)dx_i}{2(d_i \wedge d_{i+1})}$$

$$u_2 = -\frac{\Delta x_i\left(\frac{\Delta_i \wedge d_i}{\Delta x_i dx_i}\frac{dy_{i+1}}{dx_{i+1}} + \frac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}}\frac{dy_i}{dx_i}\right)}{2\frac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}}} = \frac{(d_i \wedge \Delta_i)dy_{i+1} + (d_{i+1} \wedge \Delta_i)dy_i}{2(d_i \wedge d_{i+1})}$$

RE-SKETCHING

The process of simple sketching produces a dense set of data quite smooth to the eye, but an examination of the second derivative or of the curvature (in a Euclidean context) reveals a certain erratic behavior. This microscopically erratic behavior can be eliminated, however, by the process of repeated re-sketching during each pass over the accumulated data. The process of re-sketching is as follows: On a given pass over the data and after a point has been inserted between each of the previous data points, we recalculate the nodal tangents at each of the previous data points using the previous points and the new points, throw away the new points, and recreate them using the updated nodal tangents at the previous data points. We can obviously repeat this process of re-sketching as many times as we wish per pass. Even only a few re-sketchings per pass is sufficient to markedly "iron out" the previous erratic behavior of the second derivative or curvature. What results is a pointwise interpolant that appears to have even greater smoothness than C^3. This might be expected, however, since global conic splines are infinitely smooth everywhere except at the original data points. Another convenient aspect of sketched interpolation is the fact that we need not specify any nodal tangents once and for all because of degree of freedom restrictions, as we must with the global conic splines. On the other hand, if we wish, we can specify any or all original nodal tangents once and for all. The inherent flexibility of sketched interpolation is evident. The only thing that sketched interpolation does not offer us is an analytic function describing the interpolant between each of the original data points. But since we often obtain such an interpolant only in order to generate a dense set of points on the interpolant anyway, sketching just eliminates the middleman and gives us greater flexibility and smoothness to boot. Since we compute curvature for the sketched curves in the Appendix, we develop here the formula for curvature at the midpoint of a three-point C^3 conic spline for reference. The general formula for curvature is given by

$$k = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$$

Now, recall that the second derivative of a C^3 conic spline at node i is given by

$$y_i'' = -\frac{\lambda_i A_i^3 \Delta x_i^3}{2} = -\frac{A_i^3 \Delta x_i^3}{2} \frac{2B_{i-1}(A_{i-1}\Delta y_i' \Delta x_{i-1} - B_i \Delta y_{i-1}' \Delta x_i)}{A_{i-1} A_i^2 B_i \Delta x_i^4 \Delta x_{i-1} \Delta q_{i-1}}$$

$$= \frac{A_i B_{i-1}(B_i \Delta y_{i-1}' \Delta x_i - A_{i-1} \Delta y_i' \Delta x_{i-1})}{A_{i-1} B_i \Delta x_i \Delta x_{i-1} \Delta q_{i-1}}$$

Breaking this expression down, we have

$$B_i \Delta y_{i-1}' \Delta x_i - A_{i-1} \Delta y_i' \Delta x_{i-1}$$

$$= \frac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}} \Delta x_i \frac{d_{i-1} \wedge d_i}{dx_{i-1} dx_i} - \frac{\Delta_{i-1} \wedge d_{i-1}}{\Delta x_{i-1} dx_{i-1}} \Delta x_{i-1} \frac{d_i \wedge d_{i+1}}{dx_i dx_{i+1}}$$

$$= \frac{1}{dx_{i-1} dx_i dx_{i+1}} \left((\Delta_i \wedge d_{i+1})(d_{i-1} \wedge d_i) - (\Delta_{i-1} \wedge d_{i-1})(d_i \wedge d_{i+1}) \right)$$

and

$$\frac{A_i B_{i-1}}{A_{i-1} B_i \Delta x_i \Delta x_{i-1} \Delta q_{i-1}} = \frac{\dfrac{\Delta_i \wedge d_i}{\Delta x_i dx_i} \dfrac{\Delta_{i-1} \wedge d_i}{\Delta x_{i-1} dx_i}}{\dfrac{\Delta_{i-1} \wedge d_{i-1}}{\Delta x_{i-1} dx_{i-1}} \dfrac{\Delta_i \wedge d_{i+1}}{\Delta x_i dx_{i+1}} \Delta x_i \Delta x_{i-1} \dfrac{\Delta_{i-1} \wedge \Delta_i}{\Delta x_{i-1} \Delta x_i}}$$

$$= \frac{dx_{i-1} dx_{i+1}}{dx_i^2} \frac{(\Delta_i \wedge d_i)(\Delta_{i-1} \wedge d_i)}{(\Delta_{i-1} \wedge d_{i-1})(\Delta_i \wedge d_{i+1})(\Delta_{i-1} \wedge \Delta_i)}$$

Multiplying the appropriate quantities together, we then have

$$y_i'' = \frac{1}{dx_i^3} \frac{(\Delta_i \wedge d_i)(\Delta_{i-1} \wedge d_i)}{(\Delta_{i-1} \wedge d_{i-1})(\Delta_i \wedge d_{i+1})(\Delta_{i-1} \wedge \Delta_i)} \left((\Delta_i \wedge d_{i+1})(d_{i-1} \wedge d_i) - (\Delta_{i-1} \wedge d_{i-1})(d_i \wedge d_{i+1}) \right)$$

$$= \frac{1}{dx_i^3} \frac{(\Delta_i \wedge d_i)(\Delta_{i-1} \wedge d_i)}{\Delta_{i-1} \wedge \Delta_i} \left(\frac{d_{i-1} \wedge d_i}{\Delta_{i-1} \wedge d_{i-1}} - \frac{d_i \wedge d_{i+1}}{\Delta_i \wedge d_{i+1}} \right) = \frac{1}{dx_i^3} E$$

Therefore the curvature is

$$k=\frac{\dfrac{E}{dx_i^3}}{\left(1+\dfrac{dy_i^2}{dx_i^2}\right)^{\frac{3}{2}}}=\frac{E}{(dx_i^2+dy_i^2)^{\frac{3}{2}}}$$

FINAL NOTE ON TANGENTS

We now present a more straightforward and elegant derivation of the general formula for a tangent to a conic. This derivation involves the wedge product, which we have previously defined as a real-valued binary operation on two-dimensional vectors:

$$u \wedge v = (u_1,u_2)\wedge(v_1,v_2)=u_1v_2-u_2v_1$$

The wedge product is, of course, just the oriented area of a parallelogram embraced by the two vectors. Note that area and wedge product are affine concepts rather than Euclidean ones. Using the previous definition, it is trivial to prove that

$$(u+v)\wedge w = u\wedge w + v\wedge w$$

and

$$d(u\wedge v)= du\wedge v + u\wedge dv$$

The equation of a line through two points is given by

$$\frac{y-y_i}{x-x_i}=\frac{y_j-y_i}{x_j-x_i}$$

or

$$\begin{aligned}(y-y_i)(x_j-x_i)-(x-x_i)(y_j-y_i)&=0\\ &=(x_j-x_i,y_j-y_i)\wedge(x-x_i,y-y_i)\\ &=(P_j-P_i)\wedge(P-P_i)\\ &=\Delta_{ij}\wedge(P-P_i)\end{aligned}$$

A line map may therefore be defined in terms of wedge product:

$$l_{ij}(P)=\Delta_{ij}\wedge(P-P_i)$$

from which we immediately have

$$dl_{ij}(P) = \Delta_{ij} \wedge dP$$

Recall that the equation of a conic in abridged notation is

$$a(P)b(P) + \lambda A(P)B(P) = 0$$

where a, b, A, and B are line maps and P is a point on the conic. In the case of line map a, for instance, we have

$$\begin{aligned} a(P) &= (P_{J_a} - P_{I_a}) \wedge (P - P_{I_a}) \\ &= \Delta_{I_a J_a} \wedge (P - P_{I_a}) \\ &= \Delta_a \wedge (P - P_{I_a}) \end{aligned}$$

$$da(P) = \Delta_a \wedge dP$$

Now, differentiating the equation of a conic, we have

$$\begin{aligned} &a(P)db(P) + b(P)da(P) \\ &+ \lambda(A(P)dB(P) + B(P)dA(P)) = 0 \end{aligned}$$

$$\begin{aligned} &a(P)\Delta_b \wedge dP + b(P)\Delta_a \wedge dP \\ &+ \lambda(A(P)\Delta_B \wedge dP + B(P)\Delta_A \wedge dP) = 0 \end{aligned}$$

and finally

$$T(P) \wedge dP = 0$$

where

$$T(P) = a(P)\Delta_b + b(P)\Delta_a + \lambda(A(P)\Delta_B + B(P)\Delta_A)$$

Hence, dP and $T(P)$ are parallel and $T(P)$ is a tangent vector to the conic. As before, if P_k is a point on the conic for which none of the line maps are zero, we have

$$\lambda = -\frac{a(P_k)b(P_k)}{A(P_k)B(P_k)}$$

and

$$\begin{aligned} T(P) &= a(P)\Delta_b + b(P)\Delta_a \\ &- \frac{a(P_k)b(P_k)}{A(P_k)B(P_k)}(A(P)\Delta_B + B(P)\Delta_A) \end{aligned}$$

Another tangent vector proportional to this one is therefore

$$t(P)=\frac{a(P)\Delta_b+b(P)\Delta_a}{a(P_k)b(P_k)}-\frac{A(P)\Delta_B+B(P)\Delta_A}{A(P_k)B(P_k)}$$

In the earlier section on conic differentiation, we sought the derivative or tangent at P_j and contrived to make

$$a(P_j)=A(P_j)=0$$

The tangent vector in this case simplified essentially to

$$t(P_j)=\frac{b(P_j)\Delta_a}{a(P_k)b(P_k)}-\frac{B(P_j)\Delta_A}{A(P_k)B(P_k)}$$

as it does here also. However, if one or two pairs of the five original points on the conic coalesce, however, which is to say that one or two tangents are specified for a conic through four or three points respectively, then we cannot use the simpler tangent formula and must resort to the more general one where one or two of the Δ's would be the specified tangents at the coalesced points. Note also that before two points are allowed to coalesce, they must be linked by a line map. Another important point that should be noted here is that in the case of strictly convex data, we can use the general formula for $t(P)$ to calculate the tangent at any data point after having made any reasonable selection of line maps and associated vectors once and for all! This eliminates all the special cases mentioned in the previous section on conic differentiation, making the algebra more compact and the programming more straightforward.

CONIC SECOND DERIVATIVES

In addition to obtaining tangents to local interpolating conics, one may occasionally wish also to obtain second derivatives of these interpolants. Recall that the general formula for a tangent to an interpolating conic is

$$t(P)=\frac{a(P)\Delta_b+b(P)\Delta_a}{l_k}-\frac{A(P)\Delta_B+B(P)\Delta_A}{L_k}$$

where

$$l_k=a(P_k)b(P_k) \qquad L_k=A(P_k)B(P_k)$$

Define the component functionals for an arbitrary vector v as

$$x(v)=x((v_1,v_2))=v_1 \qquad y(v)=y((v_1,v_2))=v_2$$

Since

$$t(P)\wedge\frac{dP}{dx}=0=t\wedge(1,y')=x(t)y'-y(t)$$

we have, trivially, that

$$y' = \frac{y(t)}{x(t)}$$

Differentiating

$$t \wedge (1, y') = 0$$

we have

$$dt \wedge (1, y') + t \wedge (0, dy') = 0$$

$$dt \wedge \left(1, \frac{y(t)}{x(t)}\right) + x(t) dy' = 0$$

$$-x(t)^2 dy' = dt \wedge t$$

$$-x(t)^2 y'' = \frac{dt}{dx} \wedge t$$

But

$$\frac{dt}{dx} = \frac{\frac{da}{dx}\Delta_b + \frac{db}{dx}\Delta_a}{l_k} - \frac{\frac{dA}{dx}\Delta_B + \frac{dB}{dx}\Delta_A}{L_k}$$

and since

$$\frac{dP}{dx} = (1, y') = \left(1, \frac{y(t)}{x(t)}\right) = \frac{t}{x(t)}$$

we have

$$x(t)\frac{dt}{dx} = \frac{(\Delta_a \wedge t)\Delta_b + (\Delta_b \wedge t)\Delta_a}{l_k} - \frac{(\Delta_A \wedge t)\Delta_B + (\Delta_B \wedge t)\Delta_A}{L_k}$$

Taking the wedge product of the last equation with t, we have

$$x(t)\frac{dt}{dx} \wedge t = \frac{2(\Delta_a \wedge t)(\Delta_b \wedge t)}{l_k} - \frac{2(\Delta_A \wedge t)(\Delta_B \wedge t)}{L_k}$$

from which we finally obtain the general formula for the second derivative of a local interpolating conic:

$$y'' = -\frac{2}{x(t)^3}\left[\frac{(\Delta_a \wedge t)(\Delta_b \wedge t)}{l_k} - \frac{(\Delta_A \wedge t)(\Delta_B \wedge t)}{L_k}\right]$$

From this, we immediately have a compact formula for the magnitude of the Euclidean curvature of a local interpolating conic:

$$k=\frac{2\left|(\Delta_a\wedge t)(\Delta_b\wedge t)/l_k-(\Delta_A\wedge t)(\Delta_B\wedge t)/L_k\right|}{[x(t)^2+y(t)^2]^{\frac{3}{2}}}$$

DEALING WITH INFLECTION POINTS

In order to make this development reasonably complete, it is necessary to deal with data that may have an occasional inflection point. We say that section i of the data has an inflection point if

$$w_1 w_2 < 0$$

where

$$w_1=(P_i-P_{i-1})\wedge(P_{i+1}-P_i)$$
$$w_2=(P_{i+1}-P_i)\wedge(P_{i+2}-P_{i+1})$$

What we wish to do in this section is to augment the data with additional inflection points and establish constant tangents at these inflection points. Of course, the resulting interpolant is only C^1 at these points. The problem then is twofold. We must establish a location and a tangent for each inflection point. For the method we describe, we must assume that the data is convex for a sufficient number of points both preceding and following the section in which we wish to insert the inflection point. Suppose we wish to insert an inflection point somewhere between P_i and P_{i+1}. Define the following sets.

$$\pi_L=\{P_j \mid i-3\le j\le i\}$$

$$\pi_R=\{P_j \mid i+1\le j\le i+4\}$$

Also let

$$L(P)=\textit{conic through } \pi_L\cup\{P\}$$

$$R(P)=\textit{conic through } \pi_R\cup\{P\}$$

In addition, define line maps a_L, b_L, A_L, B_L with respect to π_L and line maps a_R, b_R, A_R, B_R with respect to π_R. Now, insert a point U (as in sketching) on the leftmost section of $R(P_i)$ and a point V on the rightmost section of $L(P_{i+1})$. If we momentarily consider the conic pairs $[L(U),R(U)]$ and $[L(V),R(V)]$, it becomes obvious that a unique point W exists on the line segment UV such that $L(W)$ and $R(W)$ have the same tangent (direction) at W. The point W is given by

$$W=(1-r)U+rV \qquad 0\leq r\leq 1$$

and we seek the value of r for which

$$f(r)=t_L(W)\wedge t_R(W)=0$$

where

$$t_L(W)=\frac{\Delta_{b_L}}{b_L(W)}+\frac{\Delta_{a_L}}{a_L(W)}-\frac{\Delta_{B_L}}{B_L(W)}-\frac{\Delta_{A_L}}{A_L(W)}$$

$$t_R(W)=\frac{\Delta_{b_R}}{b_R(W)}+\frac{\Delta_{a_R}}{a_R(W)}-\frac{\Delta_{B_R}}{B_R(W)}-\frac{\Delta_{A_R}}{A_R(W)}$$

Note that we have made use of the general formula for the conic tangent here with $P_k=W$. The zero of f can be found by any number of methods. When the proper value of r has been established, W and $t(W)$ will already have been computed.

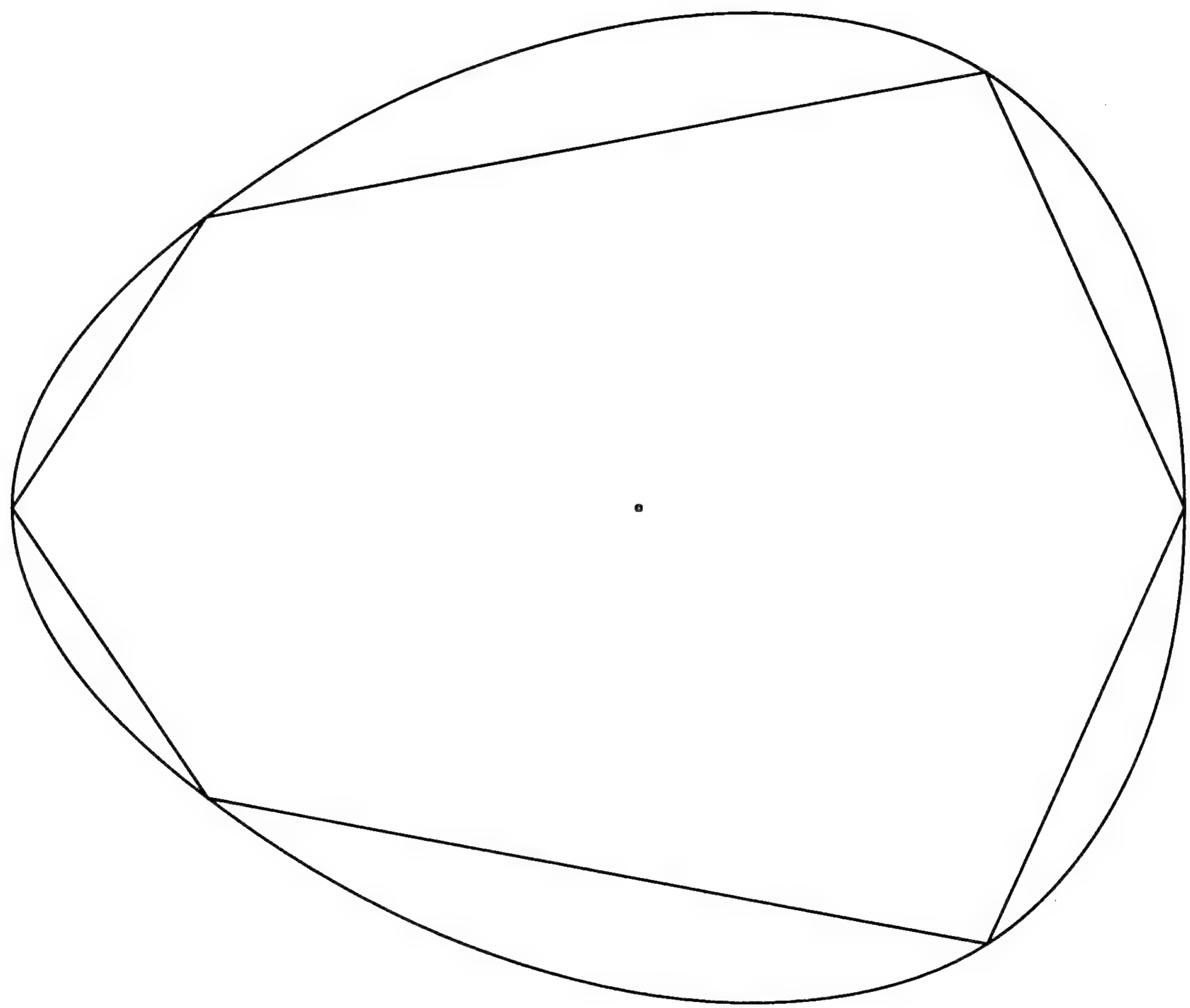
REVIEW AND CONCLUSIONS

We have shown how to obtain convexity preserving piecewise conic interpolating functions having three orders of differentiability at each data point. But, although the title of this report reflects this desired goal, the global C^3 conic spline appears (to the author, at least) to be of somewhat lesser value than other elements of this report. More important or more practical aspects of this work seem to be the simple and general formulas for conic numerical differentiation, the simple formula for oriented parabolic numerical integration which can make use of conic numerical differentiation, and sketched interpolation, which makes use of both conic numerical differentiation and the main algebraic machinery behind C^3 conic splines. The fact that sketched interpolants have apparent smoothness in excess of C^3 is due in part to the fact that none of the nodal tangents (except the ones one might wish to allow to remain fixed) are finally established until the last re-sketch of the last pass. Since sketched interpolants are not analytic functions, but merely discrete point sets, we can say nothing more precise about their added smoothness other than to refer to them as apparently C^{3+}.

The author apologizes for the lack of diagrammatic aids in this report, but he feels that verbal and algebraic precision more than make up for this lack. A picture is not always worth a thousand words, or even a hundred, if the words are sufficiently well-chosen ones. For those who disagree, however, we include in the Appendix some plots of sketched interpolants with various levels of re-sketching and accompanying polar plots of curvature with respect to the centroid of the region enclosed by the interpolant.

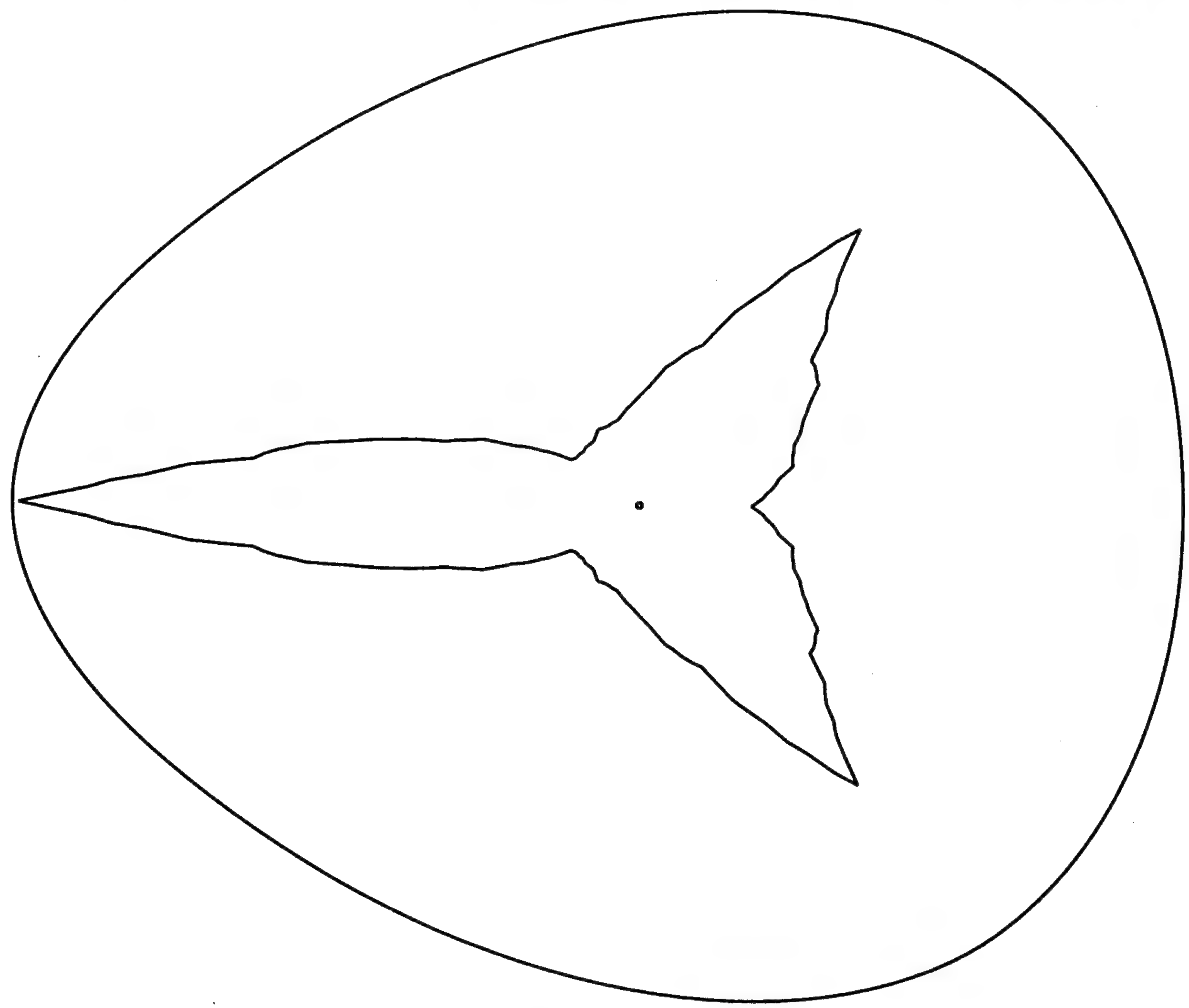
APPENDIX

Original Data and Sketched Interpolant



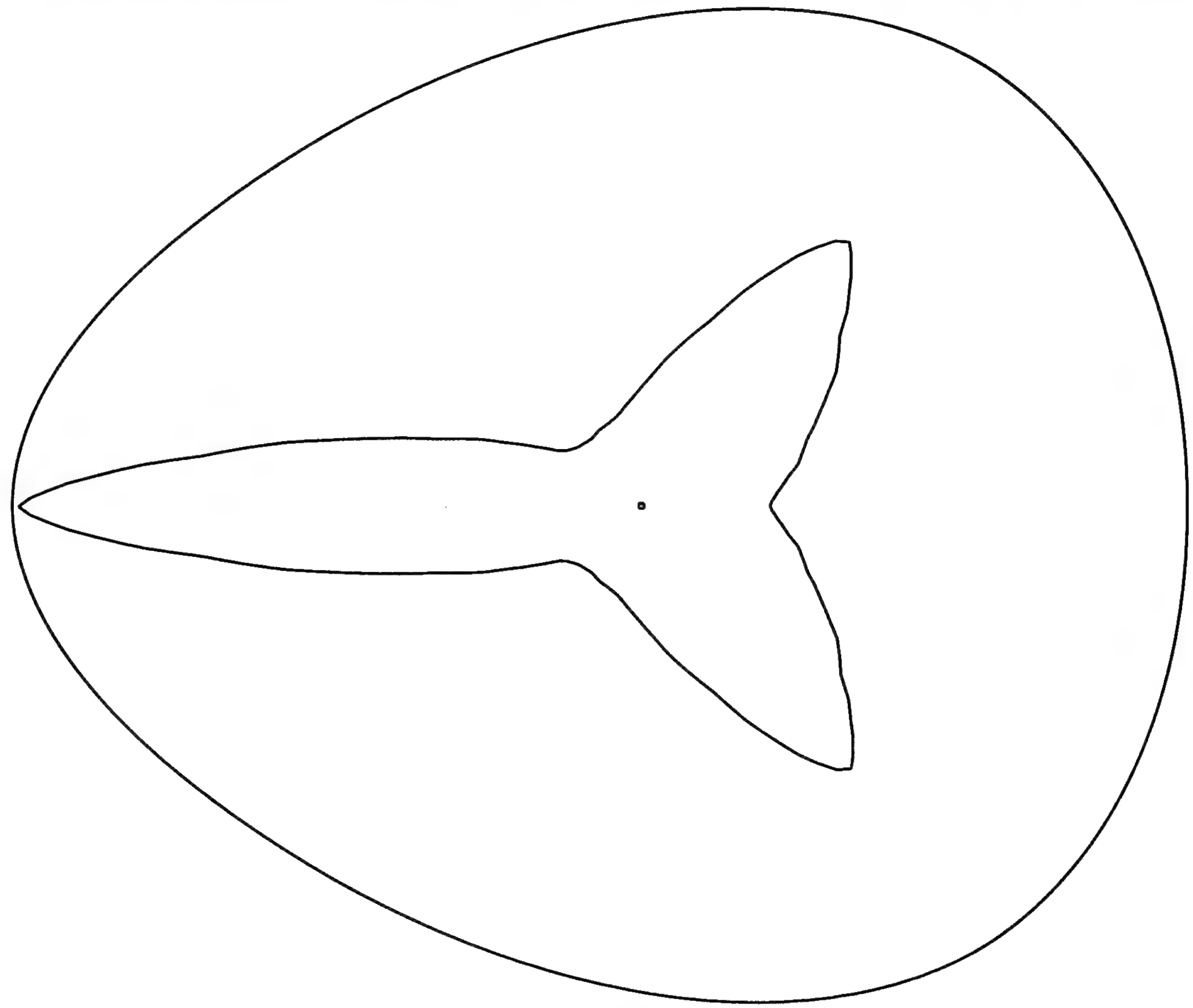
R=0

Sketched Interpolant and Curvature



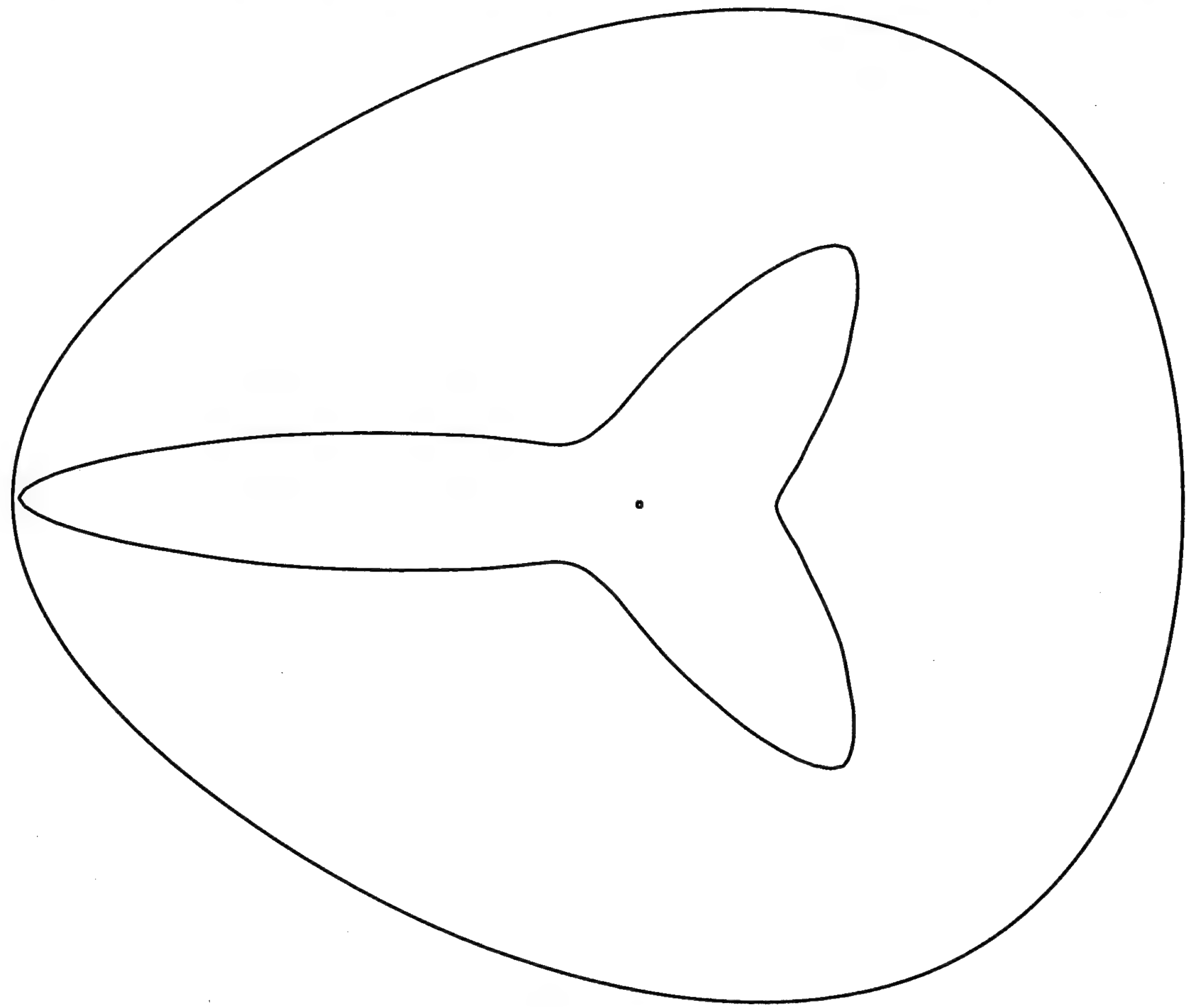
R=0

Sketched Interpolant and Curvature



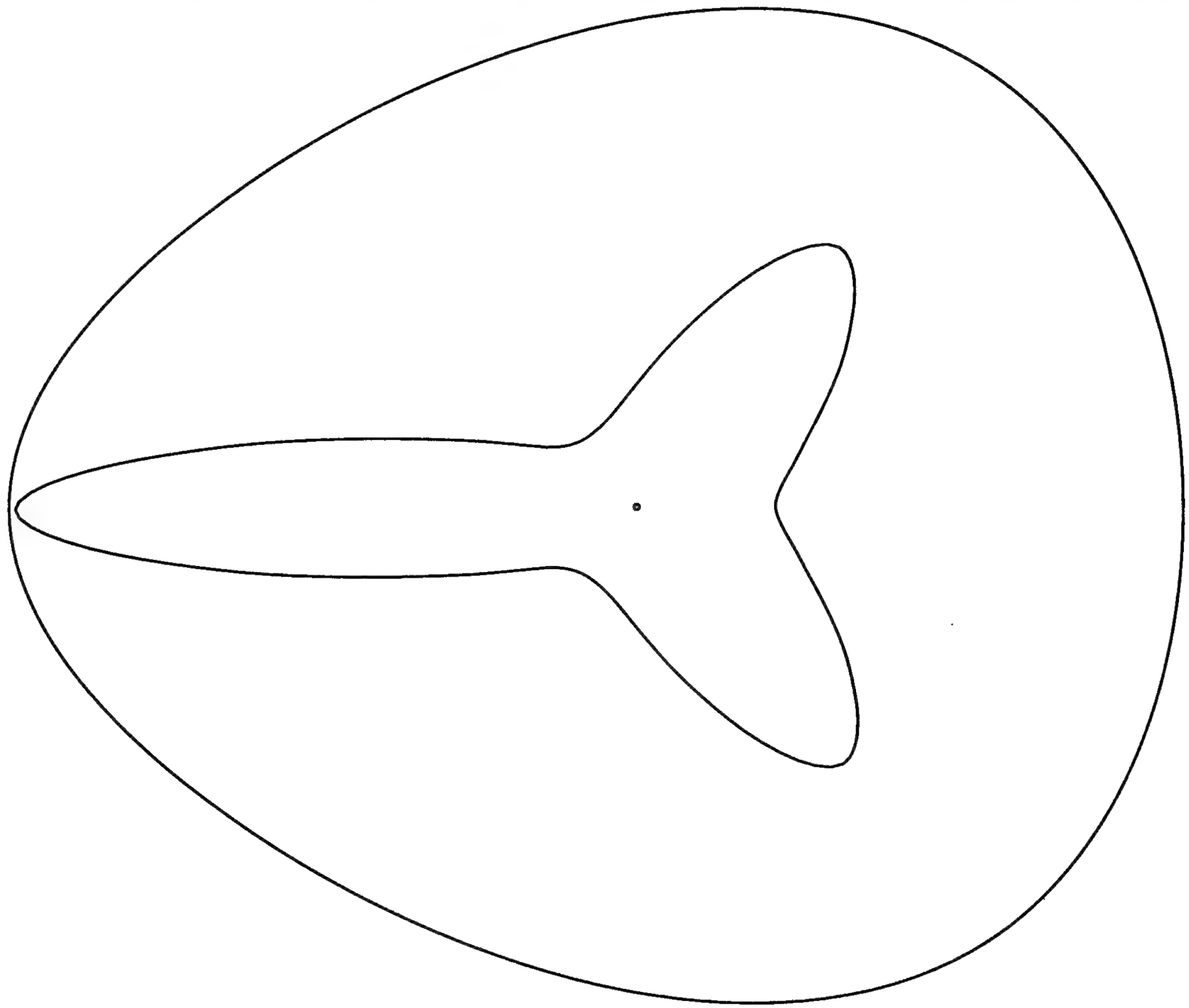
R=1

Sketched Interpolant and Curvature



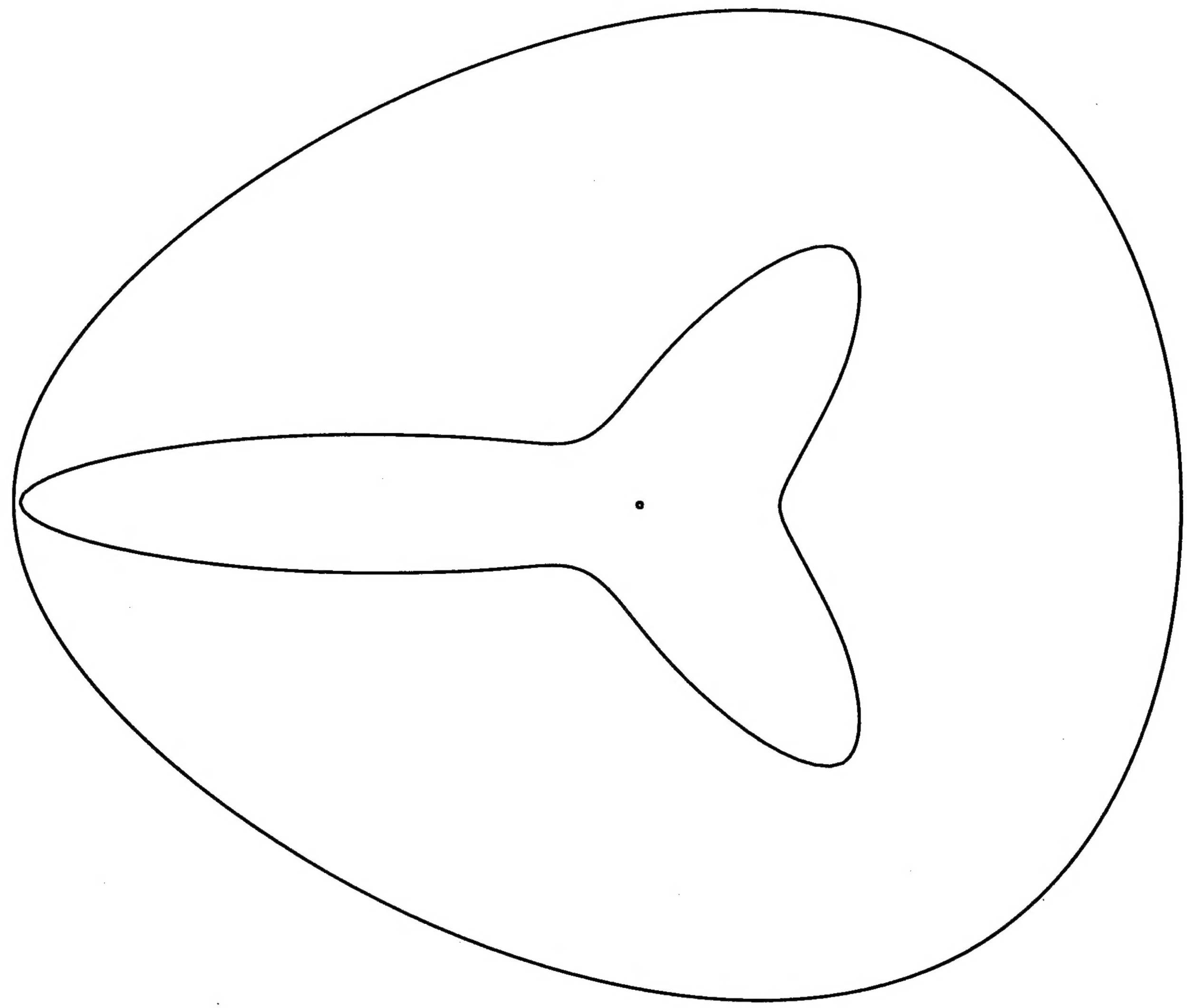
R=2

Sketched Interpolant and Curvature



R=3

Sketched Interpolant and Curvature



R=4

TECHNICAL REPORT INTERNAL DISTRIBUTION LIST

	NO. OF COPIES
CHIEF, DEVELOPMENT ENGINEERING DIVISION	
ATTN: AMSTA-AR-CCB-DA	1
-DB	1
-DC	1
-DD	1
-DE	1
CHIEF, ENGINEERING DIVISION	
ATTN: AMSTA-AR-CCB-E	1
-EA	1
-EB	1
-EC	
CHIEF, TECHNOLOGY DIVISION	
ATTN: AMSTA-AR-CCB-T	2
-TA	1
-TB	1
-TC	1
TECHNICAL LIBRARY	
ATTN: AMSTA-AR-CCB-O	5
TECHNICAL PUBLICATIONS & EDITING SECTION	
ATTN: AMSTA-AR-CCB-O	3
OPERATIONS DIRECTORATE	
ATTN: SMCWV-ODP-P	1
DIRECTOR, PROCUREMENT & CONTRACTING DIRECTORATE	
ATTN: SMCWV-PP	1
DIRECTOR, PRODUCT ASSURANCE & TEST DIRECTORATE	
ATTN: SMCWV-QA	1

NOTE: PLEASE NOTIFY DIRECTOR, BENÉT LABORATORIES, ATTN: AMSTA-AR-CCB-O OF ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST

	NO. OF COPIES
ASST SEC OF THE ARMY RESEARCH AND DEVELOPMENT ATTN: DEPT FOR SCI AND TECH THE PENTAGON WASHINGTON, D.C. 20310-0103	1
ADMINISTRATOR DEFENSE TECHNICAL INFO CENTER ATTN: DTIC-OCP (ACQUISITION GROUP) BLDG. 5, CAMERON STATION ALEXANDRIA, VA 22304-6145	2
COMMANDER U.S. ARMY ARDEC ATTN: SMCAR-AEE	1
SMCAR-AES, BLDG. 321	1
SMCAR-AET-O, BLDG. 351N	1
SMCAR-FSA	1
SMCAR-FSM-E	1
SMCAR-FSS-D, BLDG. 94	1
SMCAR-IMI-I, (STINFO) BLDG. 59 PICATINNY ARSENAL, NJ 07806-5000	2
DIRECTOR U.S. ARMY RESEARCH LABORATORY ATTN: AMSRL-DD-T, BLDG. 305 ABERDEEN PROVING GROUND, MD 21005-5066	1
DIRECTOR U.S. ARMY RESEARCH LABORATORY ATTN: AMSRL-WT-PD (DR. B. BURNS) ABERDEEN PROVING GROUND, MD 21005-5066	1
DIRECTOR U.S. MATERIEL SYSTEMS ANALYSIS ACTV ATTN: AMXSY-MP ABERDEEN PROVING GROUND, MD 21005-5071	1

	NO. OF COPIES
COMMANDER ROCK ISLAND ARSENAL ATTN: SMCRI-ENM ROCK ISLAND, IL 61299-5000	1
MIAC/CINDAS PURDUE UNIVERSITY P.O. BOX 2634 WEST LAFAYETTE, IN 47906	1
COMMANDER U.S. ARMY TANK-AUTMV R&D COMMAND ATTN: AMSTA-DDL (TECH LIBRARY) WARREN, MI 48397-5000	1
COMMANDER U.S. MILITARY ACADEMY ATTN: DEPARTMENT OF MECHANICS WEST POINT, NY 10966-1792	1
U.S. ARMY MISSILE COMMAND REDSTONE SCIENTIFIC INFO CENTER ATTN: DOCUMENTS SECTION, BLDG. 4484 REDSTONE ARSENAL, AL 35898-5241	2
COMMANDER U.S. ARMY FOREIGN SCI & TECH CENTER ATTN: DRXST-SD 220 7TH STREET, N.E. CHARLOTTESVILLE, VA 22901	1
COMMANDER U.S. ARMY LABCOM MATERIALS TECHNOLOGY LABORATORY ATTN: SLCMT-IML (TECH LIBRARY) WATERTOWN, MA 02172-0001	2
COMMANDER U.S. ARMY LABCOM, ISA ATTN: SLCIS-IM-TL 2800 POWER MILL ROAD ADELPHI, MD 20783-1145	1

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, BENÉT LABORATORIES, CCAC, U.S. ARMY TANK-AUTOMOTIVE AND ARMAMENTS COMMAND, AMSTA-AR-CCB-O, WATERVLIET, NY 12189-4050 OF ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST (CONT'D)

	NO. OF COPIES
COMMANDER U.S. ARMY RESEARCH OFFICE ATTN: CHIEF, IPO P.O. BOX 12211 RESEARCH TRIANGLE PARK, NC 27709-2211	1
DIRECTOR U.S. NAVAL RESEARCH LABORATORY ATTN: MATERIALS SCI & TECH DIV CODE 26-27 (DOC LIBRARY) WASHINGTON, D.C. 20375	1 1
WRIGHT LABORATORY ARMAMENT DIRECTORATE ATTN: WL/MNM EGLIN AFB, FL 32542-6810	1
WRIGHT LABORATORY ARMAMENT DIRECTORATE ATTN: WL/MNMF EGLIN AFB, FL 32542-6810	1

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, BENÉT LABORATORIES, CCAC, U.S. ARMY TANK-AUTOMOTIVE AND ARMAMENTS COMMAND, AMSTA-AR-CCB-O, WATERVLIET, NY 12189-4050 OF ADDRESS CHANGES.